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Floquet theory of neutrino oscillations in the earth *

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Abstract

We review the Floquet theory of linear differential equations with periodic coefficients and discuss its applications to neutrino oscillations in matter of periodically varying density. In particular, we consider parametric resonance in neutrino oscillations which can occur in such media, and discuss implications for oscillations of neutrinos traversing the earth and passing through the earth's core.

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1 Introduction

All oscillating systems are very much alike, and there are many similarities between oscillating neutrinos and, e.g., pendulums or electromagnetic circuits. In particular, neutrino oscillations in vacuum or in matter of constant density are analogous to oscillations of a simple pendulum; resonantly enhanced neutrino oscillations in a matter of monotonically varying density (the Mikheyev-Smirnov-Wolfenstein (MSW) effect [1, 2]) are similar to oscillations of two weakly coupled pendulums of slowly changing lengths [3, 4]. It is therefore natural to ask the following question: Are there any other resonance phenomena in mechanics or electromagnetism that might have analogues in neutrino physics?

One such phenomenon is parametric resonance. The parametric resonance can occur in dynamical systems with time-varying parameters when there is a certain correlation between these variations and the values of the parameters themselves. Best studied is the parametric resonance in systems with periodically varying parameters. While the periodicity makes it easier to satisfy the resonance conditions and also simplifies the analysis, it is not really necessary: parametric resonance can occur even in stochastic systems (see, e.g., [5]). In the present paper we will concentrate on the systems with periodically varying parameters.

A textbook example of a system in which the parametric resonance can occur is a pendulum with vertically oscillating point of support [6, 7]. Under certain conditions topmost, normally unstable, equilibrium point becomes stable. The pendulum can oscillate around this point in the upside-down position. Another example, familiar to everybody, is a swing, which is just a pendulum with periodically changing effective length. It is the parametric resonance that makes it possible to rock a swing.

What would be an analogue of the parametric resonance for neutrino systems? Since matter affects neutrino oscillations, periodically varying conditions can be achieved if a beam of oscillating neutrinos propagates through a medium with periodically modulated density. For certain relations between the period and amplitude of density modulation and neutrino oscillation length and mixing angle, the parametric resonance occurs, and the oscillations can be strongly enhanced. The probability of neutrino transition from one flavor state to another may become equal to unity. This phenomenon is very different from the MSW effect. Indeed, at the MSW resonance the neutrino mixing in matter becomes maximal ($\theta_m = \pi/4$) even if the vacuum mixing angle θ_0 is small. This leads to large-amplitude neutrino oscillations in a matter of constant density equal (or almost equal) to the resonance density, or to a strong flavor conversion in the case of matter density slowly varying along the neutrino path and passing through the resonance value.

The situation is quite different in the case of the parametric resonance. The mixing angle in matter does not in general become large (there is no level crossing). What happens is an amplification of the transition probability because of specific phase relationships. Thus, in the case of the parametric resonance it is the *phase* of oscillations (rather than their amplitude) that undergoes important modification. The total flavour conversion can take

place even if the mixing angles both in vacuum *and* in matter are small.

The possibility of the parametric resonance of neutrino oscillations was suggested independently in [8] and [9]. In these papers approximate solutions for sinusoidal matter density profile were found. In [9] also an exact analytic solution for the periodic step-function ("castle wall") density profile was obtained. Parametric effects in neutrino oscillations were further studied in [10] where combined action of the parametric and MSW resonances and possible consequences for solar and supernova neutrinos were considered. In this paper also the stochastic parametric resonance in neutrino oscillations was briefly discussed.

Although the parametric resonance in neutrino oscillations is certainly an interesting physical phenomenon, it requires that very special conditions be satisfied. Unfortunately, these conditions cannot be created in the laboratory because this would require either too long a baseline or neutrino propagation in a matter of too high a density. Until recently it was also unclear whether a natural object exists where these conditions can be satisfied for any known source of neutrinos. This situation has changed with a very important observation by Liu and Smirnov [11] (see also [12]), who have shown that the parametric resonance conditions can be approximately satisfied for the oscillations of atmospheric ν_μ into sterile neutrinos ν_s inside the earth. The density profile along the trajectories of neutrinos crossing the earth and passing through its core (mantle-core-mantle) is to a good approximation a piece of the periodic step-function profile, and their oscillations can be parametrically enhanced. Even though the neutrinos pass only through "1.5 periods" of density modulations (this would be exactly one period and a half if the distances neutrinos travel in the mantle and in the core were equal), the parametric effects on neutrino oscillations in the earth can be quite strong. Subsequently, it has been pointed out in [13] that the parametric resonance conditions can also be satisfied (and to even a better accuracy) for the $\nu_2 \leftrightarrow \nu_e$ oscillations in the earth in the case of the ν_e - $\nu_{\mu(\tau)}$ mixing. This, in particular, may have important implications for the solar neutrino problem. The parametric resonance in the oscillations of solar and atmospheric neutrinos in the earth was further explored in a number of papers [14, 15, 16, 17, 18, 19, 20].

In the present paper we review the Floquet theory of linear differential equations with periodic coefficients and consider its applications to neutrino oscillations and, in particular, to oscillations of neutrinos inside the earth. The paper is organized as follows. In sec. 2 we briefly review the Floquet theory and its applications to the analyses of the stability of the solutions. In sec. 3 we discuss the peculiarities of the Floquet theory in the case of the time dependent Schrödinger equations with periodic Hamiltonians. In sec. 4 we consider applications of the Floquet theory to neutrino oscillations in matter of periodic step function ("castle wall") density profile. In sec. 5 we review the implications of the parametric resonance of neutrino oscillations for neutrinos traversing the earth and passing through its core. In the last section the results are discussed and the conclusions are given.

2 Differential equations with periodic coefficients

We shall now briefly review the Floquet theory of systems of linear differential equations with periodic coefficients. More detailed discussion can be found, e.g., in [7, 21].

2.1 Preliminaries

Let us start with a few well known fact from the general theory of linear differential equations. Consider a system of n homogeneous linear differential equations

$$\dot{\psi} = \mathcal{A}(t)\psi\,, \tag{1}$$

where ψ is an n-component column vector, $\psi = (\psi_1, ..., \psi_n)^T$, $\mathcal{A}(t)$ is an $n \times n$ matrix with piecewise continuous elements, and overdot denotes differentiation with respect to t. Eq. (1) has n linearly independent continuous nontrivial solutions $\psi^{(j)}(t)$, $j = 1, ..., n$. From linearity of (1) it follows that any linear combination of the solutions is also a solution. Any set of n linearly independent solutions $\psi^{(j)}(t)$ of (1) forms the so-called fundamental set, and a matrix whose columns are $\psi^{(j)}$ is called the fundamental matrix. Given an initial condition $\psi(t_0) = \psi_0$, eq. (1) has a unique solution $\psi(t)$. Any solution of eq. (1) can be represented as a linear combination (with constant coefficients) of the solutions forming a fundamental set, or equivalently as a product of a fundamental matrix and a constant vector.

Let $\psi(t)$ be the solution of (1) with the initial condition $\psi(t_0) = \psi_0$. Let us introduce the evolution matrix $U(t, t_0)$ through the relation

$$\psi(t) = U(t, t_0)\psi_0\,. \tag{2}$$

From the definition of the evolution matrix it immediately follows that

$$U(t, t_0) = U(t, t_1)U(t_1, t_0)\,, \qquad U(t_0, t_0) = I\,, \tag{3}$$

where I is the $n \times n$ unit matrix. It is easy to check that the columns of $U(t, t_0)$ are solutions of eq. (1) with the initial conditions $\psi_i^{(j)}(0) = \delta_{ij}$. Thus $U(t, t_0)$ is a fundamental matrix, and any solution of (1) can be written in the form (2). The determinant of $U(t, t_0)$ is given by

$$det[U(t, t_0)] = \exp\left\{\int_{t_0}^{t} tr[\mathcal{A}(t')]\, dt'\right\}\,. \tag{4}$$

Eq. (4) follows from two facts: (1) the derivative of a determinant is the sum of n determinants formed by replacing elements of one row of the original matrix by their derivatives, and (2) the columns of $U(t, t_0)$ are solutions of eq. (1). Since the elements of $\mathcal{A}$ are non-singular, $det[U]$ does not vanish, i.e. the matrix $U(t, t_0)$ is non-singular. From (3) one finds

$$U(t_2, t_1)^{-1} = U(t_1, t_2)\,. \tag{5}$$

Without loss of generality, one can always choose the matrix $\mathcal{A}(t)$ to be traceless. Indeed, substituting

$$\psi(t) = e^{\alpha(t)}\psi'(t)\,, \qquad \alpha(t) = \frac{1}{n}\int^t tr[\mathcal{A}(t')]\,dt' \tag{6}$$

one finds that ψ' satisfies

$$\dot{\psi}' = \mathcal{A}'(t)\psi'\,, \qquad \mathcal{A}'(t) = \mathcal{A}(t) - \frac{1}{n}tr[\mathcal{A}(t)]\,, \tag{7}$$

i.e. $\mathcal{A}'$ is traceless. In what follows we will be always assuming that the transformation (6) has been performed, i.e. will be considering only traceless matrices $\mathcal{A}$. Eq. (4) then yields

$$det[U(t,t_0)] = 1\,. \tag{8}$$

Notice that in general $U(t,t_0)$ is not unitary.

2.2 Floquet theory

Let us now turn to the case of differential equations (1) with periodic coefficients,

$$\mathcal{A}(t+T) = \mathcal{A}(t)\,, \tag{9}$$

where T is the period. Hereafter we shall be always assuming that (9) is satisfied, without specifying this each time explicitly. From the periodicity of $\mathcal{A}(t)$ it follows that if $\psi(t)$ is a solution of (1), so is $\psi(t+T)$. Consider the solution $\psi(t)$ with the initial condition $\psi(s) = \psi_0$. We have $\psi(t+T) = U(t+T,s)\psi_0$. On the other hand, since $\psi(t+T)$ is also a solution at t, we have $\psi(t+T) = U(t,t_0)\psi(t_0+T) = U(t,t_0)U(t_0+T,s)\psi_0$. Equating these two expressions for $\psi(t+T)$ one obtains

$$U(t+T,s) = U(t,t_0)U(t_0+T,s)\,. \tag{10}$$

This is a very important property of the evolution matrix of differential equations (1) with periodic coefficients. In particular, taking $s = t_0+T$ and $s = t_0 = 0$ one obtains from (10), respectively,

$$U(t+T,t_0+T) = U(t,t_0)\,, \tag{11}$$

$$U(t+T,0) = U(t,0)\,U(T,0)\,. \tag{12}$$

The first of these equations means that the evolution matrix does not change if both its arguments are shifted by the period T (and, by induction, by any integer number k of periods). From the second equality it follows, in particular, that the matrix of evolution over k periods satisfies

$$U(kT,0) = U(T,0)^k\,. \tag{13}$$

The matrix of evolution over one period $U(T,0) \equiv U_T$ plays a very important role in the theory of differential equations with periodic coefficients; it is called the *monodromy matrix.*

As has already been pointed out, if $\psi(t)$ is a solution of (1), so is $\psi(t+T)$. This does not in general mean that $\psi(t+T) = \psi(t)$, i.e. the solutions of the equations with periodic coefficients are not in general periodic. There are, however, solutions which satisfy

$$\psi(t+T) = \sigma\psi(t)\,, \tag{14}$$

i.e. they are multiplied by a number when t is shifted by the period. Such solutions are called *normal*; they play an important role in the analysis of the stability of the solutions of eq. (1).

To analyse the properties of the normal solutions, let us show that the evolution matrix for a system of linear differential equations (1) with periodic coefficients can be written as a product of a periodic matrix and an exponential matrix. Since the monodromy matrix U_T is non-singular, it can be represented as an exponential of another matrix:

$$U_T \equiv U(T,0) = e^{\mathcal{B}T}\,. \tag{15}$$

Let us now show that the matrix $P(t) = U(t,0)e^{-\mathcal{B}t}$ is periodic. We have $P(t+T) = U(t+T,0)e^{-\mathcal{B}(t+T)} = U(t,0)U(T,0)e^{-\mathcal{B}T}e^{-\mathcal{B}t} = U(t,0)e^{-\mathcal{B}t} = P(t)$, where we have used (12) and (15). Thus the evolution matrix can be written as

$$U(t,0) = P(t)e^{\mathcal{B}t}\,, \qquad P(t+T) = P(t)\,. \tag{16}$$

From this expression it follows, in particular, that the vector χ introduced through $\psi = P(t)\chi$ satisfies the differential equation with constant coefficients:

$$\dot{\chi} = \mathcal{B}\chi\,. \tag{17}$$

Let $\phi_0^{(j)}$ be a (constant) eigenvector of the matrix $\mathcal{B}$ with the eigenvalue $\alpha_j T^{-1}$. It is then also an eigenvector of U_T with the eigenvalue $\sigma_j = e^{\alpha_j}$:

$$\mathcal{B}\phi_0^{(j)} = \frac{\alpha_j}{T}\phi_0^{(j)}\,; \qquad U_T\phi_0^{(j)} = \sigma_j\phi_0^{(j)} = e^{\alpha_i}\phi_0^{(j)}\,. \tag{18}$$

The numbers σ_j are called the characteristic numbers and α_j, the characteristic exponents. From (12), (16) and (18) it follows that any normal solution $\phi^{(j)}(t)$ of eq. (1) can be written as

$$\phi^{(j)}(t) = U(t,0)\phi_0^{(j)} = P(t)e^{\alpha_j(t/T)}\phi_0^{(j)} = P(t)\sigma_j^{t/T}\phi_0^{(j)}\,. \tag{19}$$

Thus eigenvectors of U_T give rise to normal solutions. The monodromy matrix U_T has n eigenvalues. As follows from (8), they satisfy

$$\prod_{j=1}^{n}\sigma_j = 1\,. \tag{20}$$

If all the eigenvalues σ_j are different, U_T has n linearly independent eigenvectors and therefore there are n linearly independent normal solutions $\phi^{(j)}(t)$ of system (1). Hence, they

form a fundamental set, and any solution of (1) can be written as a linear combination of the normal solutions $\phi^{(j)}(t)$ with constant coefficients. If U_T has repeated eigenvalues, the situation is more complicated and will be discussed below.

It follows from eq. (19) that the characteristic exponents (or characteristic numbers) determine the boundedness of the normal solutions, and therefore of the general solution of eq. (1). Consider, for example, the case $n = 2$. Assume first that the characteristic exponents are real. The characteristic numbers σ_1 and σ_2 are then real, too. Eq. (20) gives $\sigma_1\sigma_2 = 1$, and if $\sigma_1 \neq \sigma_2$ (i.e. they are simple eigenvalues of U_T), the absolute value of one of them is greater than one. It then follows from eq. (19) that the corresponding normal solution is unbounded. If the characteristic exponents are purely imaginary, the characteristic numbers are of modulus one, i.e. $\sigma_1 = \sigma_2^*$, $|\sigma_1| = 1$. In this case both normal solutions are bounded (the matrix $P(t)$ in (19), being continuous and periodic, is obviously bounded). If the characteristic exponents are complex, the normal solutions are bounded when the real parts of all the characteristic exponents are non-positive, while if at least one of the characteristic exponents has a positive real part, there are unbounded normal solutions. This property holds in general, i.e. for an arbitrary n.

Let us now discuss the case of repeated eigenvalues of U_T. Consider again the case $n = 2$ as an example. Assume that the eigenvalues of U_T coincide, $\sigma_1 = \sigma_2 \equiv \sigma$. The matrix U_T in this case has the general form [1]

$$U_T = \begin{pmatrix} \sigma & 0 \\ a & \sigma \end{pmatrix} . \tag{21}$$

It can be readily seen that for $a = 0$ the monodromy matrix U_T has two linearly independent and orthogonal eigenvectors corresponding to the same eigenvlue σ; they can be taken, e.g., as $\phi_0^{(1)} = (0, 1)^T$ and $\phi_0^{(2)} = (1, 0)^T$. However, for $a \neq 0$ there is only one eigenvector, $\phi_0^{(1)}$. It gives rise to the normal solution $\phi^{(1)}(t)$ through eq. (19). This normal solution can be used as one of the basis solutions $\psi^{(j)}(t)$ constituting a fundamental set, $\psi^{(1)}(t) = \phi^{(1)}(t)$. Let $\psi^{(2)}(t)$ be another solution of (1), linearly independent from $\psi^{(1)}(t)$. Then $\psi^{(1)}(t)$ and $\psi^{(2)}(t)$ form a fundamental set. Since $\psi^{(j)}(t + T)$ are also solutions, they can be written as linear combinations of $\psi^{(1)}(t)$ and $\psi^{(2)}(t)$:

$$\begin{aligned} \psi^{(1)}(t + T) &= \sigma\psi^{(1)}(t) \,, \\ \psi^{(2)}(t + T) &= a'\psi^{(1)}(t) + b'\psi^{(2)}(t) \,, \end{aligned} \tag{22}$$

Using the freedom of normalization of the vectors $\phi_0^{(j)}$, one can always choose $a' = a$. Since σ is a double root of the characteristic equation of the monodromy matrix U_T, $b' = \sigma$. [2]

[1] Another possibility would be to have the zero element $(U_T)_{21}$ instead of $(U_T)_{12}$, but the corresponding matrix can be reduced to that in (21) by a renumbering of the basis states.

[2] An easy way to see this is to consider eqs. (22) at $t = 0$ and use the explicit form (21) of the matrix U_T.

Eqs. (22) can therefore be rewritten as

$$\psi^{(1)}(t+T) = \sigma\psi^{(1)}(t)\,, \tag{23}$$
$$\psi^{(2)}(t+T) = a\psi^{(1)}(t) + \sigma\psi^{(2)}(t)\,. \tag{24}$$

Let us introduce a matrix M which relates $\psi_0^{(2)}$ and $\psi_0^{(1)} \equiv \phi_0^{(1)}$: $\psi_0^{(2)} = M\psi_0^{(1)}$ [3]. One can now find the relation between $\psi^{(2)}(t)$ and $\psi^{(1)}(t)$:

$$\psi^{(2)}(t) = W(t)\psi^{(1)}(t)\,, \qquad W(t) = U(t,0)MU(t,0)^{-1}\,. \tag{25}$$

Therefore $\psi^{(2)}(t+T) = W(t+T)\psi^{(1)}(t+T) = W(t+T)\sigma\psi^{(1)}(t)$. On the other hand, from (24) and (25), $\psi^{(2)}(t+T) = a\psi^{(1)}(t) + \sigma\psi^{(2)}(t) = [a + \sigma W(t)]\psi^{(1)}(t)$. Equating these two expressions for $\psi^{(2)}(t+T)$ one finds

$$W(t+T) = W(t) + \frac{a}{\sigma}\,. \tag{26}$$

Therefore the matrix $F(t)$ defined through

$$F(t) = W(t) - \frac{a}{\sigma}\frac{t}{T} \tag{27}$$

is periodic with the period T. From eqs. (25) and (27) we then find

$$\psi^{(2)}(t) = \left[\frac{a}{\sigma}\frac{t}{T} + F(t)\right]\psi^{(1)}(t)\,, \qquad F(t+T) = F(t)\,. \tag{28}$$

This is the result we were looking for. Together with eq. (23) it states that in the case of the double roots of the characteristic equation of U_T (i.e. in the case of coinciding characteristic exponents), a fundamental set can be chosen to consist of a normal solution and a solution which is a linear combination of a linearly growing and periodic functions multiplied by the normal solution. Thus, if $a \neq 0$, there are unbounded solutions. As follows from (28), if $a = 0$, then $\psi^{(2)}(t)$, being a product of a periodic matrix and a normal solution, is also a normal solution. Thus in this case both solutions forming a fundamental set can be chosen to be normal. This is in accord with the fact that for $a = 0$ the matrix U_T in eq. (21) has two linearly independent eigenvectors.

It is instructive to see how the linear growth of $\psi^{(2)}(t)$ arises from the general expression $\psi^{(2)}(t) = P(t)e^{\mathcal{B}t}\psi_0^{(2)}$. From (15) and (21) one finds

$$\mathcal{B}T = \ln U_T = \ln[\sigma(I+\Delta)] = \ln\sigma + \ln(I+\Delta) = \ln\sigma + \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \dots\,, \tag{29}$$

where

$$\Delta = \begin{pmatrix} 0 & 0 \\ a/\sigma & 0 \end{pmatrix}.$$

[3]The matrix M is not uniquely defined, but this is unimportant for our purposes.

Since $\Delta^2 = 0$, eq. (29) gives $\mathcal{B}T = \ln\sigma + \Delta$, hence

$$e^{\mathcal{B}t} = \sigma^{t/T} e^{\Delta(t/T)} = \sigma^{t/T}(I + \frac{t}{T}\Delta)\,. \tag{30}$$

Notice that the matrix Δ annihilates $\phi_0^{(1)}$, and therefore (30) does not contradict $\psi^{(1)}(t)$ being a normal solution. Using eq. (30) one can find a simple representation of the matrix $F(t)$ entering into eq. (28),

$$F(t) = P(t)MP(t)^{-1}\,, \tag{31}$$

from which the periodicity of $F(t)$ is obvious.

The above result can be generalized to the case $n > 2$. If the characteristic equation of the monodromy matrix U_T has repeated roots (i.e. some of the characteristic numbers coincide), a fundamental set can be chosen to consist of normal solutions and solutions which are linear combinations of polynomials in t and periodic matrices multiplied by normal solutions.

We have seen that in the case when some of the characteristic exponents have positive real parts, there are exponentially growing solutions, while if some of the characteristic exponents coincide, there are in general polynomially growing solutions. The existence of such unbounded solutions signifies instabilities due to the parametric resonance.

It should be noticed that instead of considering a system of n first order linear equations with periodic coefficients (1) one could equivalently consider one equation of the nth order. In particular, in the case $n = 2$, a general second order equation with periodic coefficients is obtained, which is called the Hill equation. A pendulum with vertically oscillating point of support mentioned in the Introduction is described by this equation. If the oscillations of the point of support are harmonic, the pendulum is described by the well known Mathieu equation. In the limit of small-amplitude oscillations of the point of support, the instability condition (the condition of exponential growth of the deviation from the equilibrium) is [6, 7]

$$\Omega \equiv \frac{2\pi}{T} = \frac{2\omega}{k}\,, \tag{32}$$

where ω is the frequency of the oscillations of the pendulum in the absence of the motion of its point of support, and k is an integer. Eq. (32) relates the frequency of the oscillations of the point of support Ω at which the parametric resonance occurs to the oscillator eigenfrequency. In general, when the amplitude of the oscillations of the point of support is not small, the parametric resonance condition depends not only on the frequency of these oscillations but also on their amplitude. In this case there are resonance regions of parameters rather than resonance values [7, 21].

In real physical systems all parameters are, of course, finite; unboundedness of certain solutions of eq. (1) with periodic coefficients just reflects the fact that in general the dynamics of real systems is only approximately described by linear equations. For large deviations from equilibrium, nonlinear effects become important and eqs. (1) have to be modified.

There are, however, cases when the solutions are always bounded even in the linear regime, and the description by linear equations can be exact. Nevertheless, parametric resonance is possible in such systems as well. One example of such a situation is given by Schrödinger equations with periodic Hamiltonians, which we discuss next.

3 Schrödinger equations with periodic Hamiltonians

If the matrix $\mathcal{A}(t)$ in eq. (1) is anti-Hermitian, the system of equations (1), (9) can be written as a Schrödinger equation with a periodic Hermitian Hamiltonian:

$$i\dot{\psi} = \mathcal{H}(t)\psi\,, \qquad \mathcal{H}(t)^\dagger = \mathcal{H}(t)\,, \qquad \mathcal{H}(t+T) = \mathcal{H}(t)\,, \tag{33}$$

where $\mathcal{H}(t) = i\mathcal{A}(t)$. In the case of constant $\mathcal{H}$, eq. (33) describes oscillations between the components of ψ characterized by $n-1$, in general different, frequencies (out of n eigenvalues of $\mathcal{H}$ only $n-1$ are independent since $tr\mathcal{H} = 0$). In particular, spin precession in a constant magnetic field or neutrino oscillations in vacuum or in matter of constant density are described by such an equation. Eq. (33) with time-dependent periodic Hamiltonians describes many physical systems, e.g., atoms in a laser field or electron paramagnetic resonance. It also describes neutrino oscillations in a medium of periodically modulated density.

Because of the Hermiticity of $\mathcal{H}(t)$, the evolution matrix $U(t, t_0)$ of eq. (33) is unitary, i.e. the norm of the vector ψ is conserved. Therefore all the solutions of eq. (33) are bounded. The parametric resonance in the systems described by eq. (33) has therefore some peculiarities, which we shall discuss below.

One consequence of the unitarity of the evolution matrix $U(t, t_0)$ is that all the characteristic exponents are purely imaginary. In addition, since the polynomial growth of the solutions is not allowed in this case, even in the case of repeated roots of the characteristic equations of the monodromy matrix U_T there are n linearly independent normal solutions which form a fundamental set. This actually directly follows from the fact that a unitary $n\times n$ matrix has exactly n linearly independent eigenvectors, irrespective of whether or not all the roots of its characteristic equations are simple.

We shall now discuss general properties of the solutions of eq. (33), using again the case $n = 2$ as an example. The results, in particular, will apply to the problem of two-flavour neutrino oscillations in a matter of periodically modulated density.

Let us start with a few general remarks about the solutions of the Shrödinger equation with time dependent (but not necessarily periodic) Hamiltonian in the case $n = 2$. First, we notice that without loss of generality the Hamiltonian $\mathcal{H}(t)$ can be considered to be real. Indeed, in the case of a complex Hamiltonian, a rephasing of the components $\psi_{1,2}$ of ψ by the factors $\exp(\pm i\beta(t)/2)$ where $\beta(t) = arg[\mathcal{H}_{12}(t)]$ transforms the Hamiltonian to the form

$$\mathcal{H}(t) = \begin{pmatrix} -A(t) & B(t) \\ B(t) & A(t) \end{pmatrix} \tag{34}$$

with real $A(t)$ and $B(t)$. Notice that this rephasing transformation preserves the trace of the Hamiltonian.

Next, we notice that the Hamiltonian $\mathcal{H}(t)$ in (34) can be written as

$$\mathcal{H}(t) = B(t)\sigma_1 - A(t)\sigma_3\,, \tag{35}$$

where σ_i are the Pauli matrices. Thus $\mathcal{H}(t)$ anticommutes with σ_2. From this fact it immediately follows that if $\psi(t) = (\psi_1(t), \psi_2(t))^T$ is a solution of the Schrödinger equation, so is $\tilde{\psi}(t) = -i\sigma_2\psi(t)^* = (-\psi_2(t)^*, \psi_1(t)^*)^T$. It is easy to see that $\psi(t)$ and $\tilde{\psi}(t)$ are orthogonal and therefore linearly independent. Thus, if one nontrivial solution of the Schrödinger equation in the case $n = 2$ is known, it automatically gives another nontrivial solution, linearly independent from the original one. The solutions ψ and $\tilde{\psi}$ form a fundamental set and therefore knowledge of a single nontrivial solution of the Schrödinger equation allows one to obtain the general solution.

We now turn to the study of the Schrödinger equation with periodic coefficients in the case $n = 2$. Since the monodromy matrix $U_T = \exp(\mathcal{B}T)$ is a unitary 2×2 matrix, it can be written as

$$U_T = Y - i\boldsymbol{\sigma}\mathbf{X} = \exp[-i(\boldsymbol{\sigma}\hat{\mathbf{X}})\Phi] \tag{36}$$

with real parameters $\mathbf{X}$ and Y (or $\hat{\mathbf{X}}$ and Φ) [4] which satisfy

$$Y^2 + \mathbf{X}^2 = 1\,, \qquad \cos\Phi = Y\,, \qquad \sin\Phi = |\mathbf{X}|\,; \qquad \hat{\mathbf{X}} \equiv \frac{\mathbf{X}}{|\mathbf{X}|}\,. \tag{37}$$

It should be noted that the form of the monodromy matrix in eq. (36) is quite general, i.e. it does not depend on the particular form of the functional dependence of $A(t)$ and $B(t)$, whereas the values of the parameters Y and $\mathbf{X} = \{X_1, X_2, X_3\}$ (or Φ and $\hat{\mathbf{X}} = \{\hat{X}_1, \hat{X}_2, \hat{X}_3\}$) are, of course, determined by this functional dependence. Using (16) one can write the evolution matrix as

$$U(t,0) = P(t)e^{-i(\boldsymbol{\sigma}\hat{\mathbf{X}})\Phi\,(t/T)}\,, \qquad P(t+T) = P(t)\,. \tag{38}$$

The matrix of evolution over an integer number of periods is

$$U(kT,0) = (U_T)^k = \exp[-i(\boldsymbol{\sigma}\hat{\mathbf{X}})k\Phi]\,. \tag{39}$$

From eqs. (15) and (36) one finds

$$\mathcal{B}T = -i(\boldsymbol{\sigma}\hat{\mathbf{X}})\Phi\,. \tag{40}$$

Since $\hat{\mathbf{X}}$ is a unit vector, the eigenvalues of the matrix $\boldsymbol{\sigma}\hat{\mathbf{X}}$ are ± 1, hence the characteristic exponents $\alpha_{1,2}$ are purely imaginary, and the characteristic numbers $\sigma_{1,2}$ are of modulus one:

$$\alpha_{1,2} = \pm i\Phi\,, \qquad \sigma_{1,2} = e^{\pm i\Phi}\,. \tag{41}$$

[4] The reality of these parameters is a consequence of the relation $\sigma_2 U(t,t_0)^*\sigma_2 = U(t,t_0)$ which, in turn, follows from the properties of the Hamiltonian $\mathcal{H}(t)^* = \mathcal{H}(t)$ and $\{\mathcal{H}(t), \sigma_2\} = 0$.

Let us find the normal solutions. The eigenvectors $\phi_0^{(1,2)}$ of the monodromy matrix coincide with the eigenvectors of the matrix $\boldsymbol{\sigma}\hat{\mathbf{X}}$; they can be written as

$$\phi_0^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1+\hat{X}_3} \\ \sqrt{1-\hat{X}_3}\, e^{i\delta} \end{pmatrix}, \qquad \phi_0^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\sqrt{1-\hat{X}_3}\, e^{-i\delta} \\ \sqrt{1+\hat{X}_3} \end{pmatrix}, \tag{42}$$

where

$$\delta = arg(\hat{X}_1 + i\hat{X}_2)\,. \tag{43}$$

Notice that $\phi_0^{(2)} = \tilde{\phi}_0^{(1)} \equiv -i\sigma_2(\phi_0^{(1)})^*$. The normal solutions are now found as $\phi^{(1,2)}(t) = U(t,0)\phi_0^{(1,2)}$, which gives

$$\phi^{(1)}(t) = P(t)e^{-i\Phi(t/T)}\phi_0^{(1)}\,, \qquad \phi^{(2)}(t) = P(t)e^{i\Phi(t/T)}\phi_0^{(2)}\,. \tag{44}$$

They form a fundamental set, which means that an arbitrary solution $\psi(t)$ can be represented as a linear combination of $\phi^{(1}(t)$ and $\phi^{(1}(t)$ with constant coefficients:

$$\psi(t) = P(t)[C_1\, e^{-i\Phi(t/T)}\phi_0^{(1)} + C_2\, e^{i\Phi(t/T)}\phi_0^{(2)}]\,. \tag{45}$$

The normal solutions (44) are the products of the periodic functions with periods T and $\tau = (2\pi/\Phi)T$. Thus, the general solution (45) describes modulated oscillations between the components of $\psi(t)$ – the parametric oscillations [20].

Let us now discuss the parametric resonance in the system under consideration. In the general case of eq. (1), the parametric resonance typically corresponds to the situations when a solution becomes unbounded, i.e. there are some values of t for which the modulus of a component of ψ can exceed any pre-assigned number, however large. A characteristic feature of the parametric resonance is that this can happen even for arbitrarily small amplitude of variations of the coefficients of eq. (1).

In the case of the systems described by eq. (33), the parametric resonance corresponds to a situation when there exist values of t for which the modulus of a component of a solution can reach maximal allowed by unitarity value, unattainable in the case of the corresponding equation with constant coefficients. This can happen even for arbitrarily small amplitude of the variations of the coefficients in eq. (33). Notice that the parametric resonance is undefined in the cases when eq. (33) with constant coefficients itself leads to maximal amplitude oscillations.

To be more specific, consider the case $n = 2$. If the Hamiltonian of the system was constant, the Schrödinger equation (33) would describe the oscillations between the components of ψ with the frequency $\omega_0 = \sqrt{A^2 + B^2}$ and amplitude $\sin^2 2\theta_0 = B^2/(A^2 + B^2)$. If $A \neq 0$, this amplitude is always less than unity. We shall now consider the case of time dependent A and B, but will be assuming that $A(t)$ never vanishes during the period of evolution of interest [5]. The transition probability in the "quasi time-independent" (adiabatic) regime

[5] In the case of neutrino oscillations in vacuum, the condition $A = 0$ corresponds to maximum mixing, while for neutrino oscillation in a matter of varying density $A(t) = 0$ corresponds to the MSW resonance.

then never exceeds $max\{B(t)^2/(A(t)^2+B(t)^2)\} < 1$. The parametric resonance occurs when there are values of t for which the modulus of a component of ψ, which was initially equal to zero, can reach the maximal allowed by unitarity value, corresponding to the transition probability equal to one. This can happen even if $max\{B(t)^2/(A(t)^2+B(t)^2)\} \ll 1$ and for arbitrarily small amplitudes of time variations of $A(t)$ and $B(t)$.

Let us find the parametric resonance condition. Assume that the initial state at $t=0$ is $\psi_0 = (1,0)^T$. Then at $t = kT$ we find

$$\psi(kT) = U(kT,0)\psi_0 = \begin{pmatrix} \cos k\Phi - i\hat{X}_3 \sin k\Phi \\ -ie^{-i\delta}\sqrt{1-\hat{X}_3^2}\,\sin k\Phi \end{pmatrix} . \tag{46}$$

We shall now show that the parametric resonance condition is [14]

$$\hat{X}_3 = 0\,. \tag{47}$$

Indeed, the transition probability reaches its maximum possible value, equal to one, when the survival probability $|\psi_1(t)|^2$ vanishes. From (46), in the case $\hat{X}_3 = 0$, one has $\psi_1(t = kT) = \cos k\Phi$. The component $\psi_1(t)$ at $t = (k+1)T$ is $\cos(k+1)\Phi$. It is easy to see that for an arbitrary nonzero value of Φ there is a value of k for which $\cos k\Phi \le 0$, $\cos(k+1)\Phi > 0$ or vice versa. Since all the solutions of eq. (33) with regular coefficients are continuous, this means that there is a value t_1, $kT \le t_1 < (k+1)T$, for which $\psi_1(t_1) = 0$ and the survival probability vanishes, i.e. the component ψ_2 saturates the unitarity limit. Thus (47) is the parametric resonance condition.

4 Neutrino oscillations in matter with "castle wall" density profile

We shall now consider applications of the Floquet theory reviewed in the preceding sections to neutrino oscillations in a matter of periodically modulated density. In particular, we shall be interested in the parametric resonance in neutrino oscillations.

Consider oscillations in a two-flavour neutrino system. The evolution of the system in the flavour eigenstate basis is described by the Schrödinger equation with the Hamiltonian (34) in which the parameters A and B are given by

$$A(t) = \frac{\Delta m^2}{4E}\cos 2\theta_0 - \frac{G_F}{\sqrt{2}}\,N(t)\,, \qquad B = \frac{\Delta m^2}{4E}\sin 2\theta_0\,, \tag{48}$$

Here G_F is the Fermi constant, E is neutrino energy, $\Delta m^2 = m_2^2 - m_1^2$, where $m_{1,2}$ are the neutrino mass eigenvalues, and θ_0 is the mixing angle in vacuum. The effective density $N(t)$

depends on the type of the neutrinos taking part in the oscillations:

$$N = \begin{cases} N_e & \text{for} \quad \nu_e \leftrightarrow \nu_{\mu,\tau} \\ 0 & \text{for} \quad \nu_\mu \leftrightarrow \nu_\tau \\ N_e - N_n/2 & \text{for} \quad \nu_e \leftrightarrow \nu_s \\ -N_n/2 & \text{for} \quad \nu_{\mu,\tau} \leftrightarrow \nu_s\,. \end{cases} \tag{49}$$

Here N_e and N_n are the electron and neutron number densities, respectively. For transitions between antineutrinos one should substitute $-N$ for N in eq. (48). If overall matter density and/or chemical composition varies along the neutrino path, the effective density N depends on the neutrino coordinate t. The instantaneous oscillation length $l_m(t)$ and mixing angle $\theta_m(t)$ in matter are given by

$$l_m(t) = \pi/\omega(t)\,, \qquad \sin 2\theta_m(t) = B/\omega(t)\,, \qquad \omega(t) \equiv \sqrt{B^2 + A(t)^2}\,. \tag{50}$$

The MSW resonance corresponds to $A(t_{res}) = 0$, $\sin 2\theta_m(t_{res}) = 1$.

For the parametric resonance to occur, the exact shape of the matter density profile is not very important; what is important is that the change in the density be synchronized in a certain way with the change of the oscillation phase. In particular, in [8, 9] the case of the sinusoidal density profile was considered in which the neutrino evolution equation reduces to a modified Mathieu equation. In [9] the parametric resonance was also considered for neutrino oscillations in a matter with a periodic step function density profile, which allows a very simple exact analytic solution. This solution was studied in detail in [14, 20]. Here we shall review this solution and its main features.

Consider the case when the effective density $N(t)$ (and therefore $A(t)$) is a periodic step function:

$$N(t) = \begin{cases} N_1\,, & 0 \le t < T_1 \\ N_2\,, & T_1 \le t < T_1 + T_2 \end{cases}$$

$$N(t+T) = N(t)\,, \quad T = T_1 + T_2\,. \tag{51}$$

Here N_1 and N_2 are constants. We shall call this the “castle wall” density profile (see fig. 2). The function $A(t)$ is expressed by similar formula with constants A_1 and A_2. Thus, the Hamiltonian $\mathcal{H}(t)$ is also a periodic function of time with the period T. Let us denote

$$\delta = \frac{\Delta m^2}{4E}\,, \qquad V_i = \frac{G_F}{\sqrt{2}}\,N_i \qquad (i = 1,\ 2)\,. \tag{52}$$

In this notation

$$A_i = \cos 2\theta_0\,\delta - V_i\,, \qquad B = \sin 2\theta_0\,\delta\,, \qquad \omega_i = \sqrt{(\cos 2\theta_0\,\delta - V_i)^2 + (\sin 2\theta_0\,\delta)^2}\,. \tag{53}$$

Any instant of time in the evolution of the neutrino system belongs to one of the two kinds of the time intervals:

$$\begin{aligned} &(1): \quad 0 + nT \le t < T_1 + nT \\ &(2): \quad T_1 + nT \le t < T_1 + T_2 + nT\,, \quad n = 0,\ 1,\ 2, ... \end{aligned} \tag{54}$$

In either of these time intervals the Hamiltonian $\mathcal{H}$ is a constant matrix which we denote $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let us define the evolution matrices for the intervals of time $(0, T_1)$ and $(T_1, T_1 + T_2)$:

$$U_1 = \exp(-i\mathcal{H}_1 T_1)\,, \qquad U_2 = \exp(-i\mathcal{H}_2 T_2) \tag{55}$$

The monodromy matrix is then

$$U_T = U_2 U_1\,. \tag{56}$$

Let us introduce the unit vectors

$$\begin{aligned}\mathbf{n}_1 &= \frac{1}{\omega_1}(B,\ 0,\ \ -A_1) = (\sin 2\theta_1,\ 0,\ \ -\cos 2\theta_1)\,,\\ \mathbf{n}_2 &= \frac{1}{\omega_2}(B,\ 0,\ \ -A_2) = (\sin 2\theta_2,\ 0,\ \ -\cos 2\theta_2)\,,\end{aligned} \tag{57}$$

where $\theta_{1,2}$ are the mixing angles in matter at densities N_1 and N_2: $\theta_1 = \theta_m(N_1)$, $\theta_2 = \theta_m(N_2)$. Then one can write

$$\mathcal{H}_i = \omega_i(\boldsymbol{\sigma}\mathbf{n}_i)\,. \tag{58}$$

Using eqs. (55) - (58) one can obtain the monodromy matrix in the form (36) with the parameters Y and $\mathbf{X}$ given by

$$Y = c_1 c_2 - (\mathbf{n_1 n_2}) s_1 s_2\,, \tag{59}$$

$$\mathbf{X} = s_1 c_2\, \mathbf{n}_1 + s_2 c_1\, \mathbf{n}_2 - s_1 s_2\, (\mathbf{n}_1 \times \mathbf{n}_2)\,, \tag{60}$$

where we have used the notation

$$s_i = \sin\phi_i\,, \quad c_i = \cos\phi_i\,, \quad \phi_i = \omega_i T_i \quad (i = 1,\ 2)\,. \tag{61}$$

Notice that the difference of the neutrino eigenenergies in a matter of density N_i is $2\omega_i$, so that $2\phi_1$ and $2\phi_2$ are the oscillations phases acquired over the intervals T_1 and T_2. The evolution matrix for k periods (k=0, 1, 2,...) is given by eq. (39).

The vector $\mathbf{X}$ can be written in components as

$$\mathbf{X} = \{(s_1 c_2 \sin 2\theta_1 + s_2 c_1 \sin 2\theta_2)\,, \quad -s_1 s_2 \sin(2\theta_1 - 2\theta_2)\,, \quad -(s_1 c_2 \cos 2\theta_1 + s_2 c_1 \cos 2\theta_2)\}\ . \tag{62}$$

Eqs. (36) - (39) and (59) - (62) give the exact solution of the evolution equation for any instant of time that is an integer multiple of the period T. In order to obtain the solution for $kT < t < (k+1)T$ one has to evolve the solution at $t = kT$ by applying the evolution matrix

$$U_1(t,\ kT) = \exp[-i\mathcal{H}_1 \cdot (t - kT)] \tag{63}$$

for $kT < t < kT + T_1$ or

$$U_2(t,\ kT + T_1) U_1 = \exp[-i\mathcal{H}_2 \cdot (t - kT - T_1)] \exp[-i\mathcal{H}_1 T_1] \tag{64}$$

for $kT + T_1 \le t < (k+1)T$, with $\mathcal{H}_{1,2}$ given by eq. (58).

4.1 Parametric resonance

Assume that the initial neutrino state at $t = 0$ is a flavor eigenstate ν_a. The probability of finding another flavor eigenstate ν_b at a time $t > 0$ (transition probability) is then $P(\nu_a \to \nu_b,\ t) = |U_{21}(t,0)|^2$. As pointed out in sec. 3, the evolution of neutrino system in a matter of periodically varying density has a character of parametric oscillations – modulated oscillations characterized by two periods, T and $\tau = (2\pi/\Phi)T$. As can bee seen from eq. (46), the transition probability after passing k periods of density modulation is [14]

$$P(\nu_a \to \nu_b,\ t = kT) = (1 - \hat{X}_3^2) \sin^2 \Phi_p \,, \qquad \Phi_p = k\Phi \,, \tag{65}$$

where Φ was defined in (37). Notice that this expression is valid for any periodic matter density profile, irrespective of its shape. The values of $\hat{X}_3$ and Φ, of course, depend on this shape. For neutrino oscillations in matter with the "castle wall" density profile, eq. (65) corresponds to the evolution over an even number of layers of constant density. The transition probability after passing an odd number of alternating layers, which can be considered as k periods plus one additional layer of density N_1 (the corresponding evolution time $t = kT + T_1$), is also given by eq. (65), the only difference being that the phase is now [20]

$$\Phi_p = k\Phi + \varphi \tag{66}$$

with

$$\begin{aligned} &\sin\varphi = s_1 \sin 2\theta_1 / \sqrt{1 - \hat{X}_3^2}\,, \\ &\cos\varphi = (s_1 \sin 2\theta_1 Y + s_2 \sin 2\theta_2) / \sqrt{\mathbf{X}^2 - X_3^2} \end{aligned} \tag{67}$$

Eqs. (65) - (67) give the transition probability at the borders of the layers.

The parametric resonance occurs when the depth of the parametric oscillations described by (65) becomes equal to unity, i.e. when $\hat{X}_3 = 0$. This coincides with the condition (47) found in sec. 3. For the case of the "castle wall" density profile under consideration it reads [14] (see eq. (62)):

$$X_3 \equiv -(s_1 c_2 \cos 2\theta_1 + s_2 c_1 \cos 2\theta_2) = 0\,. \tag{68}$$

As follows from (65), the maximum transition probability $P = 1$ can be achieved at the borders of the layers provided that

$$\Phi_p = \frac{\pi}{2} + n\pi\,, \qquad n = 0, 1, 2, ... \tag{69}$$

The parametric resonance condition (68) can be realized in two different ways. One possibility is that both terms on the right hand side of eq. (68) vanish. This requires $c_1 = c_2 = 0$ [8 – 14], or [6]

$$\phi_1 = \frac{\pi}{2} + k'\pi\,, \qquad \phi_2 = \frac{\pi}{2} + k''\pi\,, \qquad k', k'' = 0, 1, 2, ... \tag{70}$$

[6]In refs. [8, 9, 10] these conditions were derived for the particular case $k' = k''$, which includes the most important principal resonance with $k' = k'' = 0$.

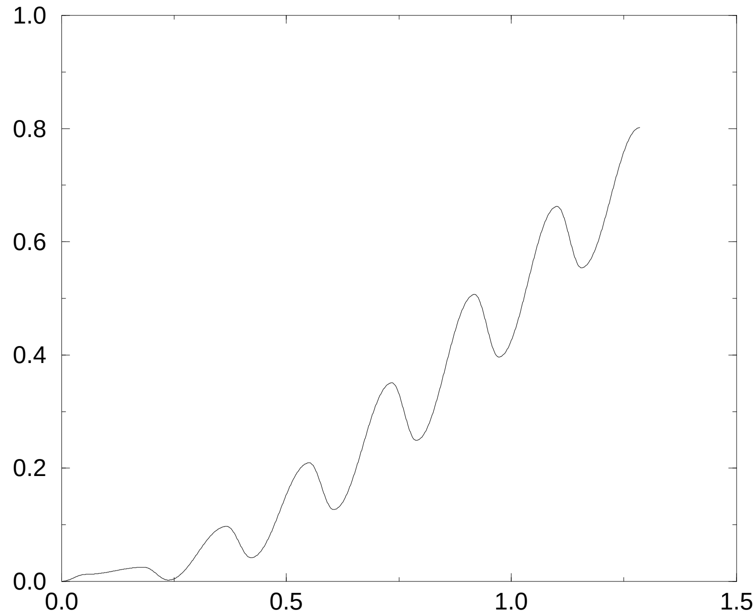

Figure 1: Coordinate dependence of the neutrino flavor transition probability P in a matter with the "castle wall" density profile. $\sin^2 2\theta_0 = 0.01$, $\delta = 10^{-12}$ eV, $V_1 = 10^{-13}$ eV, $V_2 = 6.33 \times 10^{-13}$ eV, $T_1 = 5.4 \times 10^{-2}$, $T_2 = 0.1296$, all distances are in units of $R = 3.23 \times 10^{13}$ eV^{-1}.

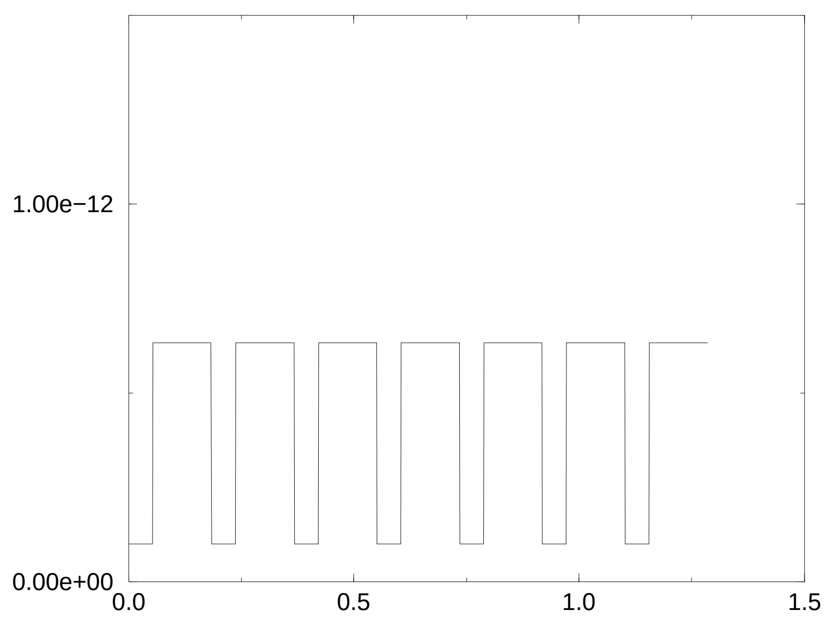

Figure 2: Coordinate dependence of the matter-induced neutrino potential $[G_F/\sqrt{2}\times$ (density profile)] for the case shown in fig. 1.

Notice that another option, $s_1 = s_2 = 0$, leads to a trivial case $\mathbf{X} = 0$, $Y = \pm 1$, in which the monodromy matrix coincides (up to the sign) with the unit matrix, and the transition probability on the borders of the layers vanishes [7]. Another possible realization of the parametric resonance condition is when neither of the terms on the right hand side of eq. (68) vanishes but they exactly cancel each other.

We shall consider now the realization (70) of the parametric resonance condition (68) (the second realization will be illustrated by a numerical example in fig. 7). At the resonance, the transition probability for the evolution over k periods of density modulation takes a simple form

$$P(\nu_a \to \nu_b,\ t = kT) = \sin^2[k(2\theta_2 - 2\theta_1)]\,. \tag{71}$$

Let us first assume that the densities N_1, N_2 are either both below the MSW resonance density N_{MSW}, which is determined from $G_F N_{MSW}/\sqrt{2} = \cos 2\theta_0\, \delta$, or they are both above it. This means that the mixing angles in matter $\theta_{1,2}$ satisfy $\theta_{1,2} < \pi/4$ or $\theta_{1,2} > \pi/4$, respectively. It is easy to see that in this case the difference $2\theta_2 - 2\theta_1$ is always farther away from $\pi/2$ than either $2\theta_1$ or $2\theta_2$. Therefore in this case the transition probability for evolution over one period cannot exceed the maximal transition probabilities in matter of constant density equal to either N_1 or N_2, namely, $\sin^2 2\theta_1$ or $\sin^2 2\theta_2$. However, the parametric resonance does lead to an important gain. In a medium of constant density N_i the transition probability can never exceed $\sin^2 2\theta_i$, no matter how long the distance that neutrinos travel. On the contrary, in the matter with "castle wall" density profile, if the parametric resonance conditions (70) are satisfied, the transition probability can become large provided neutrinos travel a large enough distance. It can be seen from (71) that the transition probability can become quite sizeable even for small $\sin^2 2\theta_1$ and $\sin^2 2\theta_2$ (i.e., in terms of the parameters of the Hamiltonian (34), for small $max\{B^2/(A(t)^2 + B^2)\}$). This is illustrated in figs. 2 and 3 for the case $N_1, N_2 < N_{MSW}$ (the transition probability in the case $N_1, N_2 > N_{MSW}$ has a similar behavior). The number of periods neutrinos have to pass in order to experience a complete (or almost complete) conversion is

$$k \simeq \frac{\pi}{4(\theta_1 - \theta_2)}\,. \tag{72}$$

It is instructive to consider the limit of small density variations, $|N_1 - N_2| \ll N_1$. In terms of the analogy with a pendulum with vertically oscillating point of support, it corresponds to the limit of the small amplitude of these vertical oscillations. In this limit $\theta_1 \simeq \theta_2$, and eq. (68) reduces to $\sin(\phi_1 + \phi_2) = 0$, or $\phi_1 + \phi_2 = k\pi$. Since $\phi_i = \omega_i T_i$, this condition can be written as

$$\Omega \equiv \frac{2\pi}{T} = \frac{2\omega}{k}\,, \qquad \text{where} \qquad \omega \equiv \omega_1 \frac{T_1}{T} + \omega_2 \frac{T_2}{T}\,. \tag{73}$$

[7] We do not consider the trivial cases of the MSW resonance for which $X_3 = 0$ because $\cos 2\theta_i = 0$ and $s_i = \pm 1$, $i = 1$ or 2, or $\cos 2\theta_1 = \cos 2\theta_2 = 0$. These cases correspond to $A(t_{res}) = 0$.

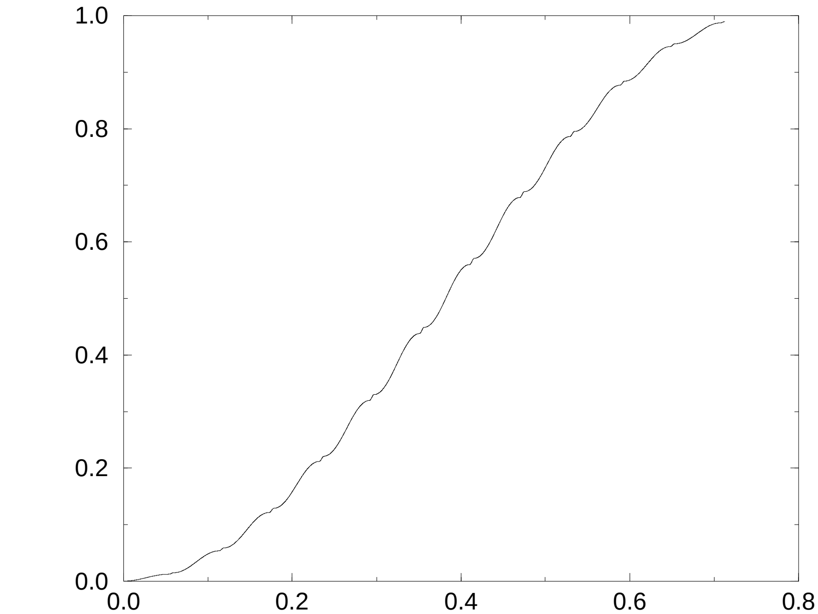

Figure 3: Same as in fig. 1 but for $\delta = 10^{-12}$ eV, $V_2 = 10^{-11}$ eV, $T_1 = 5.4\times10^{-2}$, $T_2 = 5.4\times10^{-3}$.

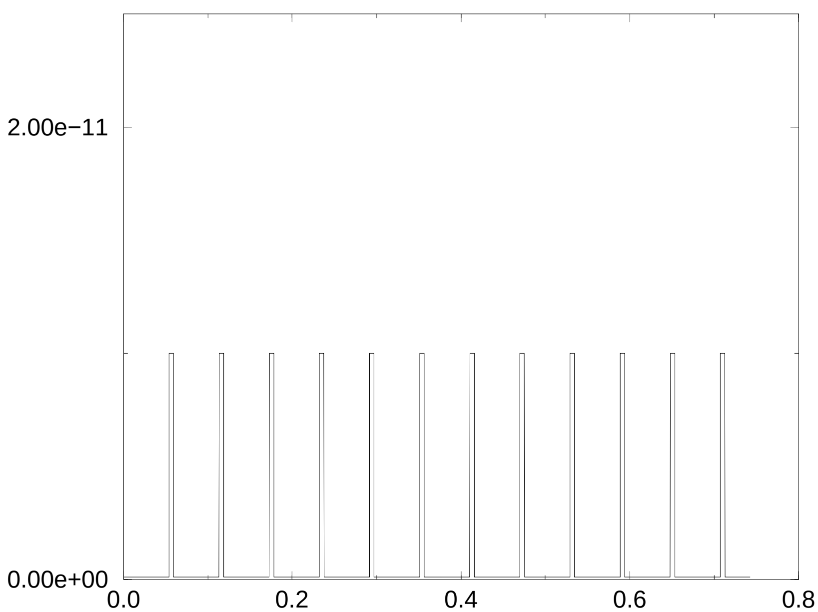

Figure 4: Coordinate dependence of the matter-induced neutrino potential for the case shown in fig. 3.

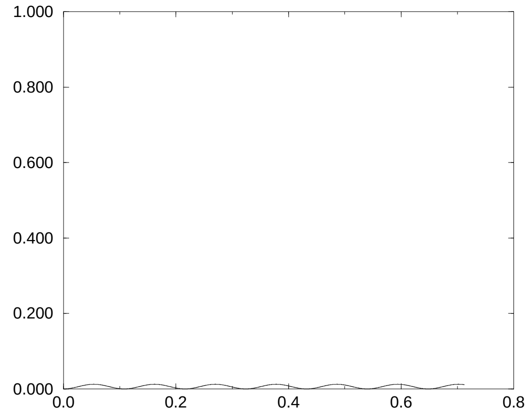

Figure 5: Same as in fig. 3 but for $V_2 = V_1$ ($V(t) = V_1 = const$).

This coincides with the familiar parametric resonance condition in the case of the small-amplitude variations of the parameter of the system, eq. (32). It is important to notice that the condition (73) does not depend on the amplitude of the density modulation, $N_1 - N_2$. This illustrates the point that we emphasized in sec. 3 – the parametric resonance can occur even for arbitrarily small amplitude of the variations of the parameters of the system. Of course, the smaller this amplitude, the longer the evolution time for the total conversion.

Consider now the case $N_1 < N_{MSW} < N_2$ ($\theta_1 < \pi/4 < \theta_2$). The transition probability over n periods at the parametric resonance is again given by eq. (71). However in this case, for $\theta_2 > \pi/4 + \theta_1/2$ (which is always satisfied for small mixing in matter), one has $\sin^2(2\theta_2 - 2\theta_1) > \sin^2 2\theta_1, \sin^2 2\theta_2$. This means that *even for the time interval equal to one period of matter density modulation the transition probability exceeds the maximal probabilities of oscillations in matter of constant densities* N_1 *and* N_2. The case $N_1 < N_{MSW} < N_2$ is illustrated in figs. 3, 6 and 7.

Figs. 3 and 4 show the importance of the phase relationships in the case of the parametric resonance. In these figures the coordinate dependence of the transition probability and matter density profile are shown for a specific case in which conditions (70) are fulfilled. It can be seen from these figures that the probability increase during the time intervals T_2, which correspond to the effective matter density N_2, is very small, and, in addition, in this case $T_2 \ll T_1$. One could therefore conclude that the evolution during these very narrow intervals is unimportant. However, this conclusion is wrong: if one removes the "spikes" in the matter density profile of fig. 4, i.e. replaces it by the profile $N(t) = N_1 = const$, the resulting transition probability will be very small at all times (fig. 5).

Further examples of the parametrically enhanced neutrino oscillations can be found in figs. 6 and 7. Figures 1, 3 and 6 correspond to the realization (70) ($c_1 = c_2 = 0$) of the parametric resonance condition (68); fig. 7 illustrates the realization in which the two terms in X_3 cancel each other.

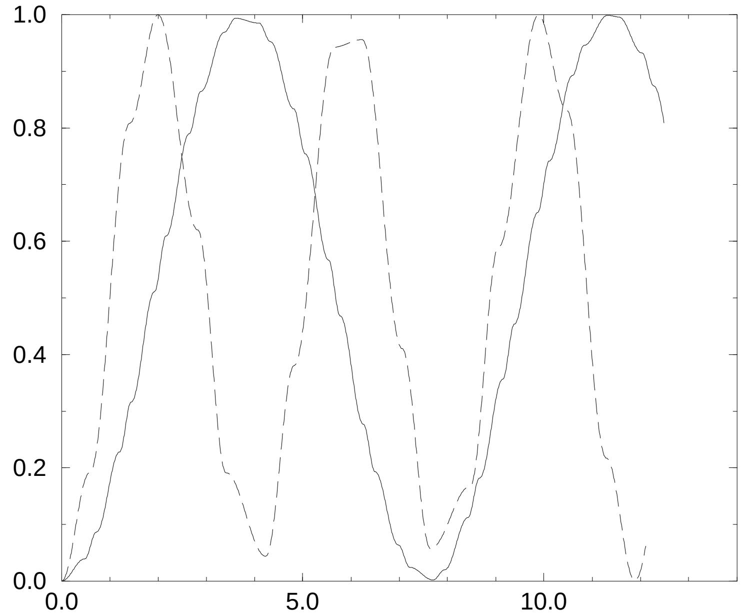

Figure 6: Solid curve: Coordinate dependence of the transition probability P for the case of total conversion over 5 periods of density modulation (10 layers). Dashed curve: the same for the case of total conversion over 3 layers. The kinks correspond to the borders of the layers of different densities. The curves were plotted for the realization (70) ($c_1 = c_2 = 0$) of the parametric resonance condition.

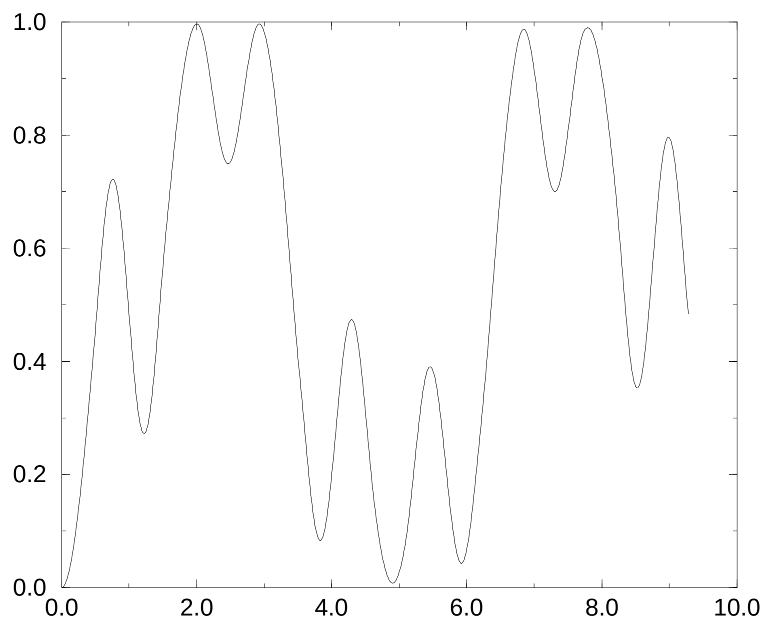

Figure 7: Same as in fig. 6 but for a case when the parametric resonance condition is realized through the cancellation of the two terms in eq. (32); total conversion is achieved over 3 layers.

5 Parametric resonance in neutrino oscillations in the earth

5.1 Evolution of oscillating neutrinos in the earth

The earth consists of two main structures, the mantle and the core, which can to a very good approximation be considered as layers of constant density. We shall consider neutrino oscillations in the earth in this two-layer approximation. Neutrinos coming to the detector from the lower hemisphere of the earth at zenith angles Θ in the range $\cos\Theta = (-1) \div (-0.837)$ (nadir angle $\Theta_n \equiv 180^\circ - \Theta \leq 33.17^\circ$) traverse the earth's mantle, core and then again mantle, i.e. three layers of constant density with the third layer being identical to the first one. Therefore such neutrinos experience a periodic "castle wall" potential, and their oscillations can be parametrically enhanced. Although the neutrinos propagate in this case only through three layers ("1.5 periods" of density modulation), the parametric enhancement of the transition probability can be very strong.

The evolution matrix in this case is $U = U_1 U_2 U_1$. It can be parametrized in a form similar to that in eq. (36):

$$U = Z - i\boldsymbol{\sigma}\mathbf{W}\,, \qquad Z^2 + \mathbf{W}^2 = 1\,. \tag{74}$$

The matrix U describes the evolution of an arbitrary initial state and therefore contains all the information relevant for neutrino oscillations. In particular, the probabilities of the neutrino flavor oscillations P and of $\nu_2 \leftrightarrow \nu_e$ oscillations P_{2e} (relevant for the oscillations of solar and supernova neutrinos inside the earth) are given by [14] [8]

$$P = W_1^2 + W_2^2\,, \qquad P_{2e} = \sin^2\theta_0 + W_1(W_1\cos 2\theta_0 + W_3 \sin 2\theta_0)\,. \tag{75}$$

Equivalently, the probability P can be described by eqs. (65) – (67) with $k = 1$.

We have now to identify the effective densities N_1 and N_2 with the average matter densities N_m and N_c in the earth's mantle and core, respectively; similarly, we change the notation $V_{1,2} \to V_{m,c}$, $\phi_{1,2} \to \phi_{m,c}$ and $\theta_{1,2} \to \theta_{m,c}$.

In the two-layer approximation, the parameters Z, $\mathbf{W}$ have a very simple form [14]:

$$Z = 2\cos\phi_m\, Y - \cos\phi_c\,, \tag{76}$$

$$\mathbf{W} = \left(2\sin\phi_m \sin 2\theta_m\, Y + \sin\phi_c \sin 2\theta_c\,,\ 0\,,\ -\left(2\sin\phi_m\cos 2\theta_m\, Y + \sin\phi_c \cos 2\theta_c\right)\right)\,. \tag{77}$$

Here the vector $\mathbf{W}$ was written in components; the parameter Y was defined in (59). If the parametric resonance condition (68) is satisfied through the realization (70), the neutrino flavor transition probability takes the value [11, 12]

$$P = \sin^2(2\theta_c - 4\theta_m)\,, \tag{78}$$

[8] Different but equivalent expressions can also be found in [22, 23, 13].

whereas the probability of the $\nu_2 \leftrightarrow \nu_e$ transitions is [13]

$$P_{2e} = \sin^2(2\theta_c - 4\theta_m + \theta_0)\,. \tag{79}$$

These probabilities can be close to unity (the arguments of the sines close to $\pi/2$) even if the amplitudes of neutrino oscillations in the mantle, $\sin^2 2\theta_m$, and in the core, $\sin^2 2\theta_c$, are rather small. This can happen if the neutrino energy lies in the range $E_c < E < E_m$, where E_m and E_c are the values of the energy that correspond to the MSW resonance in the mantle and in the core of the earth. This condition is equivalent to $N_m < N_{MSW} < N_c$. In the case of small mixing angle MSW solution of the solar neutrino problem, $\sin^2 2\theta_0 < 10^{-2}$, and P_{2e} practically coincides with P unless both probabilities are very small.

The trajectories of neutrinos traversing the earth are determined by their nadir angle $\Theta_n = 180° - \Theta$. The distances R_m and R_c that neutrinos travel in the mantle (each layer) and in the core are given by

$$R_m = R\left(\cos\Theta_n - \sqrt{r^2/R^2 - \sin^2\Theta_n}\right)\,, \qquad R_c = 2R\sqrt{r^2/R^2 - \sin^2\Theta_n}\,. \tag{80}$$

Here $R = 6371$ km is the earth's radius and $r = 3486$ km is the radius of the core. The matter density in the mantle of the earth ranges from 2.7 g/cm^3 at the surface to 5.5 g/cm^3 at the bottom, and that in the core ranges from 9.9 to 12.5 g/cm^3 (see, e.g., [24]). The electron number fraction Y_e is close to 1/2 both in the mantle and in the core. Taking the average matter densities in the mantle and core to be 4.5 and 11.5 g/cm^2 respectively, one finds for the $\nu_e \leftrightarrow \nu_{\mu,\tau}$ oscillations involving only active neutrinos the following values of V_m and V_c: $V_m = 8.58 \times 10^{-14}$ eV, $V_c = 2.19 \times 10^{-13}$ eV. For transitions involving sterile neutrinos $\nu_e \leftrightarrow \nu_s$ and $\nu_{\mu,\tau} \leftrightarrow \nu_s$, these parameters are a factor of two smaller.

5.2 Parametric resonance conditions for neutrino oscillations in the earth

If the parametric resonance conditions (70) are satisfied, strong parametric enhancement of the oscillations of core crossing neutrinos in the earth can occur [11, 12, 13, 14, 15, 16], see fig. 8. In some cases, condition (69) can also be fulfilled, and the parametric resonance leads to a complete flavour conversion for neutrinos traversing the earth.

We shall now discuss the resonance conditions (70). The phases ϕ_m and ϕ_c depend on the neutrino parameters Δm^2, θ_0 and E and also on the distances R_m and R_c that the neutrinos travel in the mantle and in the core. The path lengths R_m and R_c vary with the nadir angle; however, as can be seen from (80), their changes are correlated and they cannot take arbitrary values. Therefore if for some values of the neutrino parameters a value of the nadir angle Θ_n exists for which, for example, the first condition in eq. (70) is satisfied, it is not obvious if at the same value of Θ_n the second condition will be satisfied as well. In other words, it is not clear if the realization (70) of the parametric resonance condition (68) is

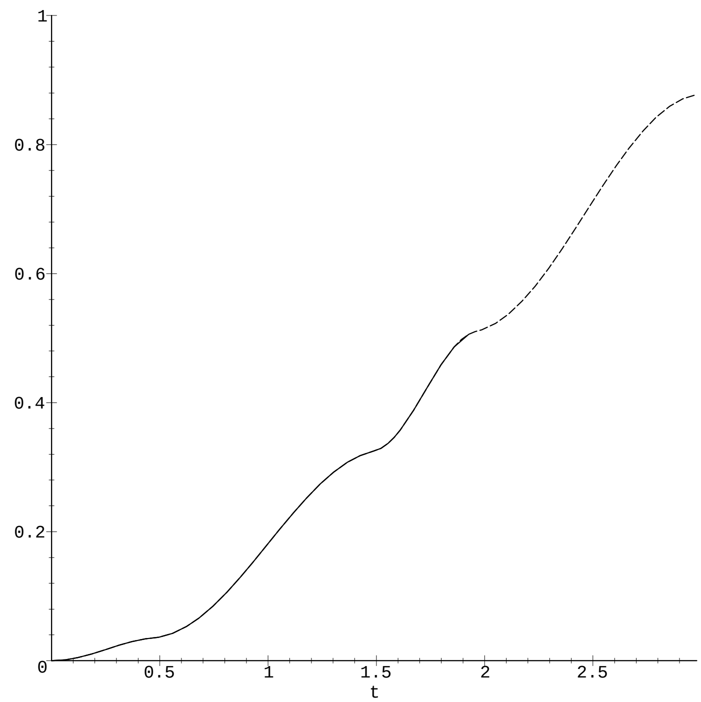

Figure 8: Solid curve: transition probability P for $\nu_e \leftrightarrow \nu_{\mu,\tau}$ oscillations in the earth as a function of the distance t (measured in units of the earth's radius R) along the neutrino trajectory. $\delta \equiv \Delta m^2/4E = 1.8\times 10^{-13}$ eV, $\sin^2 2\theta_0 = 0.01$, $\Theta_n = 11.5°$. Dashed curve: the same for a hypothetical case of neutrino propagation over full two periods of density modulation ($t_{max} = 2(R_m + R_c)$).

possible for neutrino oscillations in the earth for at least one set of the neutrino parameters Δm^2, θ_0 and E. However, as was shown in [13, 14], not only the parametric resonance conditions (70) are satisfied (or approximately satisfied) for a rather wide range of the nadir angles covering the earth's core, they are fulfilled for the ranges of neutrino parameters which are of interest for the neutrino oscillations solutions of the solar and atmospheric neutrino problems. In particular, the conditions for the principal resonance ($k' = k'' = 0$) are satisfied to a good accuracy for $\sin^2 2\theta_0 \lesssim 0.1$, $\delta \simeq (1.1 \div 1.9)\times 10^{-13}$ eV^2, which includes the ranges relevant for the small mixing angle MSW solution of the solar neutrino problem and for the subdominant $\nu_\mu \leftrightarrow \nu_e$ and $\nu_e \leftrightarrow \nu_\tau$ oscillations of atmospheric neutrinos.

The fact that the parametric resonance conditions (70) can be satisfied so well for neutrino oscillations in the earth is rather surprising. It is a consequence of a number of remarkable numerical coincidences. It has been known for some time that the potentials V_m and V_c corresponding to the matter densities in the mantle and core, the inverse radius of the earth R^{-1}, and typical values of $\delta \equiv \Delta m^2/4E$ of interest for solar and atmospheric neutrinos, are all of the same order of magnitude – $(3\times 10^{-14} - 3\times 10^{-13})$ eV (see, e.g., [25, 11, 26]). It

is this surprising coincidence that makes appreciable earth matter effects on the oscillations of solar and atmospheric neutrinos possible. However, for the parametric resonance to take place, a coincidence by an order of magnitude is not sufficient: the conditions (70) have to be satisfied at least within a 50% accuracy [14]. This is exactly what takes place. In addition, in a wide range of the nadir angles Θ_n, with changing Θ_n the value of the parameter δ at which the resonance conditions (70) are satisfied slightly changes, but the fulfillment of these conditions is not destroyed.

Even more surprising is the fact that the second realization of the parametric resonance condition (68), the one in which the two terms in X_3 cancel each other, is also possible for neutrino oscillations in the earth [19]. This requires a very subtle tuning between the values of neutrino parameters θ_0 and $\delta = \Delta m^2/4E$ on one hand and the effective matter densities N_m and N_c and neutrino pathlengths R_m and R_c in the mantle and in the core on the other. This looks very contrived, and yet turns out to be possible. Moreover, for a number of the values of the neutrino parameters, the condition (69) can also be fulfilled and a complete flavour conversion for neutrinos crossing the earth is is possible [19].

6 Discussion and conclusion

We have reviewed the Floquet theory of linear differential equations with periodic coefficients and discussed its applications for neutrino oscillations in matter of periodically modulated density. In particular, in the case of two-flavour oscillations, we have shown that the evolution of the system takes a form of parametric oscillations – modulated oscillations characterized by two periods, the period T of density modulations and $\tau = (2\pi/\Phi)T$, where $\pm i\Phi$ are the characteristic exponents. We have also discussed the parametric resonance in neutrino oscillations and have shown that, irrespective of the shape of the periodic density modulations, the parametric resonance condition is $X_3 = 0$, where X_3 is one of the parameters that determine the monodromy matrix given in eq. (36). We have also reviewed the exact solution in the case of 2-flavour neutrino oscillations in a matter of periodic step-function density profile, obtained in [9, 14], and discussed its relations with the Floquet theory. This solutions allows one to find explicit formulas for the parameters entering into the monodromy matrix, including the characteristic exponents and X_3. We discussed the implications of this exact solution for the oscillations of neutrinos inside the earth, the density profile of which can to a very good accuracy be approximated by a piece of the periodic step-function profile. We concentrated on possible parametric resonance effects in neutrino oscillations in the earth.

Besides being an interesting physical phenomenon, the parametric resonance in neutrino oscillations can provide us with an important additional information about neutrino properties. Therefore experimental observation of this effect would be of considerable interest. The prospects for experimental observation of the parametric resonance in oscillations of solar and atmospheric neutrinos traversing the earth were discussed in [17, 18]. To a large

extent, these prospects depend on the values of some of neutrino parameters which are not well known yet. The bottom line is that such an experimental observation is difficult but may be possible.

Parametric enhancement may lead to noticeable effects in oscillations of supernova neutrinos in the earth, resulting in characteristic distortions of the spectra of neutrinos crossing the earth's core [27]. A sufficiently accurate measurement of the supernova neutrino spectrum would, however, require a relatively close supernova ($L \leq 10$ kpc).

An interesting possibility to study the parametric effects in neutrino oscillations would be very long baseline experiments with intense neutrino beams produced at neutrino factories (for discussions of neutrino oscillations experiments at neutrino factories see, e.g., [28]). For baselines larger than approximately 10700 km, neutrinos would cross the earth's core, and it would be possible to probe the parametric resonance effects in $\nu_e \leftrightarrow \nu_{\mu(\tau)}$ oscillations. Notice, however, that the present feasibility studies concentrate on relatively short baselines, a few thousand km [28]. In addition, the expected average energies of neutrino beams ($E \geq 20$ GeV) in the currently discussed experiments are somewhat higher than what would be desirable in order to study the parametric effects.

As we have seen, observing the parametric resonance in oscillations of solar, atmospheric or supernova neutrinos in the earth or in experiments with neutrino factories is not an easy task. Can one create the necessary matter density profile and observe the parametric resonance in neutrino oscillations in the laboratory (i.e. short-baseline) experiments? Unfortunately, the answer to this question seems to be negative: this would require either too long a baseline or neutrino propagation in a matter of too high a density (see [17, 18] for details). One can conclude that the sole presently known object where the parametric resonance in neutrino oscillations can take place is our planet, as was first pointed out in [11, 12].

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